Poisson error for angular correlation functions

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This short document describes the Poisson error bars as used in `athena`.

The Landy-Szalay estimator

The Landy-Szalay (LS) estimator of the angular (cross-)correlation function between two data sets i and j at angular separation θ is

$$w_{ij}^{\mathrm{LS}}(\theta) = A_{ij} \frac{\langle D_i D_j \rangle_\theta}{\langle RR \rangle_\theta} - \frac{B_i \langle D_i R \rangle_\theta + B_j \langle D_j R \rangle_\theta}{\langle RR \rangle_\theta} + 1. \tag{1}$$

The quantities in this expression are defined as follows. $\langle XY \rangle_\theta$ is the number of pairs between object catalogues X and Y in an angular bin around θ. D_i is the i^{th} data set (i.e., galaxy position catalogue), and R denotes the random catalogue.

Originally, the LS esimator was defined using a random catalogue with the same number of objects than the data set. However, to reduce Poisson noise, one usually creates a random catalogue with many more objects. Therefore, the pair counts in the LS estimator have to be scaled by the total number of pairs in the corresponding catalogues. This is achieved with the constants A and B: For the number of pairs from a single catalogue, i.e. for the auto-correlation, `athena` counts each pair only once. Therefore, if N_i denotes the number of objects in data set i, the total number of pairs $\langle D_i D_i \rangle_{\mathrm{tot}}$ is $N_i(N_i - 1)/2$, whereas the total number of pairs $\langle D_i D_j \rangle_{\mathrm{tot}}$ for $i \neq j$ is $N_i N_j$, which is approximatetly twice the number of pairs $\langle D_i D_i \rangle_{\mathrm{tot}}$.

The constants A and B are therefore given by

$$\begin{aligned} A_{ij} &= \begin{cases} \frac{N_R(N_R-1)}{2N_i N_j} & \text{for} \quad i \neq j \\ \frac{N_R(N_R-1)}{N_i(N_i-1)} & \text{for} \quad i = j \end{cases} \\ B_i &= \frac{N_R(N_R-1)}{2N_R N_i} = \frac{N_R - 1}{2N_i}. \end{aligned} \tag{2}$$

The number of random objects is N_R.

Poisson error of the LS estimator

The Poisson error of (1) is obtained using Gaussian error propagation. First, we note that

$$\Delta(\langle D_i D_j \rangle_\theta) = \sqrt{\langle D_i D_j \rangle_\theta}. \tag{3}$$

Then

$$\left|\frac{\partial w_{ij}^{\mathrm{LS}}}{\partial\langle D_iD_j\rangle_\theta}\Delta(\langle D_iD_j\rangle_\theta)\right|^2 = A_{ij}^2\frac{\langle D_iD_j\rangle_\theta}{\langle RR\rangle_\theta^2};$$

$$\left|\frac{\partial w_{ij}^{\mathrm{LS}}}{\partial\langle D_iR\rangle_\theta}\Delta(\langle D_iR\rangle_\theta)\right|^2 = \begin{cases} B_i^2\frac{\langle D_iR\rangle_\theta}{\langle RR\rangle_\theta^2} & \text{for} \quad i\neq j \\ (2B_i)^2\frac{\langle D_iR\rangle_\theta}{\langle RR\rangle_\theta^2} & \text{for} \quad i=j \end{cases}$$

$$\begin{aligned}\left|\frac{\partial w_{ij}^{\mathrm{LS}}}{\partial\langle RR\rangle_\theta}\Delta(\langle RR\rangle_\theta)\right|^2 &= \Big(A_{ij}\langle D_iD_j\rangle_\theta - B_i\langle D_iR\rangle_\theta - B_j\langle D_jR\rangle_\theta\Big)^2\frac{\langle RR\rangle_\theta}{\langle RR\rangle_\theta^4} \\ &= \frac{(w_{ij}^{\mathrm{LS}}-1)^2}{\langle RR\rangle_\theta}. \qquad (4)\end{aligned}$$

To get the total Poisson error, we sum all individual terms,

$$\Delta\left(w_{ij}^{\mathrm{LS}}\right)^2 = \frac{A_{ij}^2\langle D_iD_j\rangle_\theta}{\langle RR\rangle_\theta^2} + (1+\delta_{ij})\frac{B_i^2\langle D_iR\rangle_\theta + B_j^2\langle D_jR\rangle_\theta}{\langle RR\rangle_\theta^2} + \frac{(w_{ij}^{\mathrm{LS}}-1)^2}{\langle RR\rangle_\theta}, \qquad (5)$$

with Kronecker's delta δ_{ij}.

The Hamilton estimator

The Hamilton estimator of the angular (cross-)correlation function is

$$w_{ij}^{\mathrm{H}} = 2(1+\delta_{ij})\frac{\langle D_iD_j\rangle_\theta\langle RR\rangle_\theta}{\langle D_iR\rangle_\theta\langle D_jR\rangle_\theta} - 1. \qquad (6)$$

The prefactor 4 (2) of $w_{ij}^{\mathrm{H}}+1$ for $i=j$ ($i\neq j$) accounts for the fact that the number of pairs for auto- and cross-correlations differ by a factor of two.

Poisson error of the Hamilton estimator

We obtain the Poisson error, by first noting that

$$\left|\frac{\partial w_{ij}^{\mathrm{H}}}{\partial X}\Delta(X)\right|^2 = \frac{\left(w_{ij}^{\mathrm{H}}+1\right)^2}{X} \qquad (7)$$

for $X=\langle D_iD_j\rangle_\theta, X=\langle RR\rangle_\theta$ and $X=\langle D_iR\rangle_\theta$ if $i\neq j$. Moreover, for the auto-correlation,

$$\left|\frac{\partial w_{ii}^{\mathrm{H}}}{\partial\langle D_iR\rangle_\theta}\Delta(\langle D_iR\rangle_\theta)\right|^2 = 4\frac{\left(w_{ii}^{\mathrm{H}}+1\right)^2}{\langle D_iR\rangle_\theta} \qquad (8)$$

Therefore,

$$\Delta\left(w_{ij}^{\mathrm{H}}\right)^2 = \left(w_{ij}^{\mathrm{H}}+1\right)^2\left[\frac{1}{\langle D_iD_j\rangle_\theta} + (1+\delta_{ij})\left(\frac{1}{\langle D_iR\rangle_\theta} + \frac{1}{\langle D_jR\rangle_\theta}\right) + \frac{1}{\langle RR\rangle_\theta}\right]. \qquad (9)$$